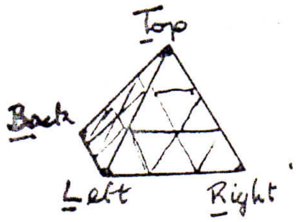
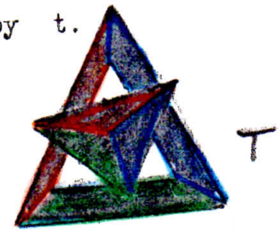
1. A Notation for Moves

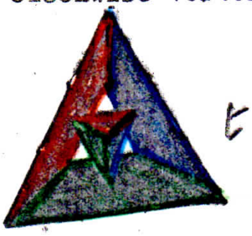
In order to examine more closely the operation of the pyraminx and to work out good sequences of moves, it is necessary to be able to write down moves in some suitable notation and to have a systematic way of telling what pattern or "state" of the pyraminx a given sequence produces. The diagrams and coloured patterns that have been used so far have served well for description, but are not so effective for analysis

Here is a notation for giving a sequence of moves which is clear, accurate and can be written concisely. Place the pyraminx in the way it has usually been illustrated, with an underneath face, a face to the front and one on either side sloping to a vertex at the back.

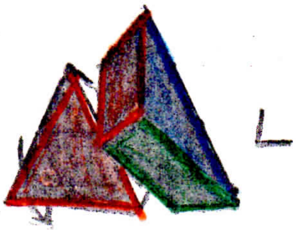


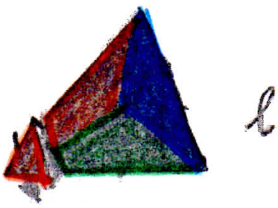
Label the vertices as shown. Now denote a full twist of the top in a clockwise direction by the letter T and a clockwise vertex twist of the top by t.





Similarly a full clockwise twist of the left vertex will be denoted by L and a vertex twist of the same left vertex by ℓ .



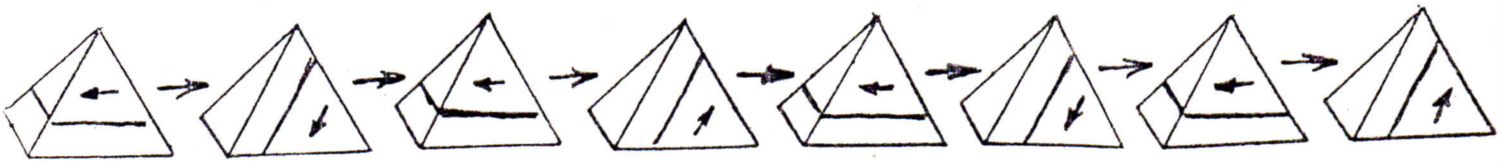


For right twists the symbols are R and r and for twists of the back vertex B and b, naturally! Twists in an anticlockwise direction will be denotes by T', t', L', ℓ', R', r', B' and b' respectively.

Using this notation, [redacted] a quick sequence of moves leading to the Jewel pattern may be written:

T R T R' T R T R',

eight moves in all. In the diagram method these would be shown as:



but the sequence of letters is a precisely equivalent statement.

As a second example, consider the sequence of 20 moves giving Joseph's Coat:

R L R L R L R L R L T B T B T B T B T B .

If we think of the letters representing the moves as being "multiplied together" when they are placed side-by-side, then this sequence could be written briefly as:

$$(RL)^5(TB)^5 .$$

A final example of the notation is provided by Pharoah's Cup, which can be obtained with the sequence:

L' R L R B R T' B' T' B T' B' (T't) .

If the final pair T't of this sequence is regarded as a single "slice" move, then this is a 13 move sequence.

The notation for moves is concise and helpful for recording sequences of moves. Sometimes it is also useful for spotting small shortcuts. Notice that a repeated move can be replaced by a single one, RR = R' for instance, while a pair of moves such as T T' is effectively no move at all. These may seem obvious statements to make in terms of the notation, but when one is turning the pyraminx around and about it is possible to miss the fact that such pairs of moves are being made. For example a quick and common way to make Doors is R L' R' L followed by a turn of the pyraminx and then another similar sequence. ~~left hand leading instead of the right~~. Analysis with the written notation shows that the second sequence is in fact L B' L' B, and putting the two together gives not an eight move sequence, but a seven move sequence, namely: R L' R' L' B' L' B. The final L of the sequence along with the first L of the second sequence together give a single L'.

The Doors 7-move sequence can be coupled with a 7-move sequence for Doubles, namely R' T R T R' T R to give Chariot in only 13 moves: R' T R T R' T R' L' R' L' B' L' B .

There are things for which we as yet have no mechanism, however, in spite of the notation. It is not obvious what is the end effect of a given sequence of moves, nor is it clear when two sequences of moves lead to the same pattern. For example, who would have thought that

$$(TR'L)^3 \qquad\qquad (9 \text{ moves})$$

and R' L R L' T L' T' L B' L B L' R L' R' L T' L T L' B L' B' L (24 moves) lead to the same pattern? The second sequence (three Cat's Paws) is in some senses the more logical way of ariving at the King's Treasure Chest, but it is certainly not the most efficient!

2. Recording a Pattern

It is easy to record the positions of the vertex and middle blocks. We do it by noting the composite move needed to bring the blocks from their "pure" positions to the given one without affecting any other blocks. Thus we shall write

$$J_t,\ J_\ell,\ J_r,\ J_b$$

for clockwise Jewel moves about the top, left, right and back vertices respectively.

As it turns out, it will be convenient to modify slightly the notation for vertex moves. Write

$$V_t,\ V_\ell,\ V_r,\ V_b$$

for t, ℓ, r and b, and

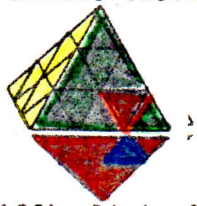
$$V_t',\ V_\ell',\ V_r',\ V_b'$$

for t', ... , b' respectively. Similarly J_t', ... , J_b' will be the anticlockwise Jewel moves.

Try up to here

With this symbolism the pattern Joseph's Coat can be denoted by the product of moves: $J_tJ_\ell J_rJ_b$.
The effect of a single anticlockwise turn of the middle block on the right may be denoted by

$$J_r'V_r\ ,$$

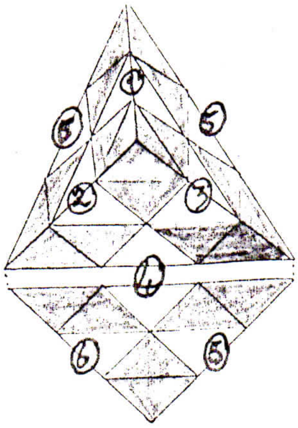


and so on.

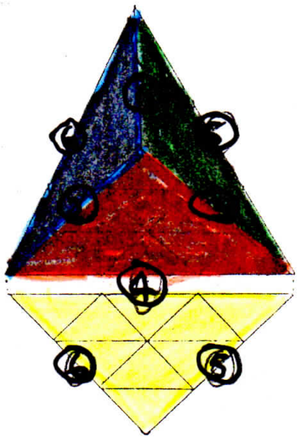
The positions of the edge blocks are more difficult to describe, firstly because there are as many as six blocks which can be rearranged amongst themselves, and secondly because they can be "flipped" in pairs, and so the parity, or "flippedness", of the blocks must be taken into account as well as their position.

Now that we are dealing with the mathematical properties of the Pyraminx, which will be true no matter what colours or surfaces are used for decoration, it is desirable to have a notation which is independent of colour. There are systems for doing this on the basis of the "top/left/right/back" labelling we have already introduced, but they all contain the seeds of confusion with the notation for moves, and therefore a "customised" notation may do the job better. For this we simply number the edge positions 1, 2, ... , 6 and then set up a glossary as in the following table. The colours given in the first column refer to the illustrated sample colouring. The reader's colouring may be different, in which case the second column may be filled in to replace the first. Notice that the order in which the colours in each pair are mentioned is important.

Colouring in Sample	Reader's Colouring	Position Number	Position Symbol after flip
Blue/Green		1	$\bar{1}$
Red/Blue		2	$\bar{2}$
Green/Red		3	$\bar{3}$
Yellow/Red		4	$\bar{4}$
Green/Yellow		5	$\bar{5}$
Yellow/Blue		6	$\bar{6}$



Numbering of
Edge Positions



Sample
Colouring

Suppose the pyraminx is given a move T. Ignoring what happens to vertices and middles, for the moment, the effect of T on the edge blocks is given by the following table:

Block in position	moves to	position
1	$\longrightarrow$	3
2	$\longrightarrow$	1
3	$\longrightarrow$	2
4	$\longrightarrow$	4
5	$\longrightarrow$	5
6	$\longrightarrow$	6

Edge moves under T

Of course "block in position 4 moves to position 4" means that this block is unaffected by T. The same is true for the blocks in positions 5 and 6.

In this example the parity (flippedness) of the blocks takes care of itself since the first named face of position 1 (the blue face of the blue/green position) moves to the first named face of position 3 (the green face of the green/red position), and so on.

We abbreviate the above table to the symbol

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 4 & 5 & 6 \end{pmatrix}$$

or, even more briefly, to (132). The latter ~~exceeding~~ very concise "short form" is called the <u>cyclic notation</u> and means:

"Position 1 moves to the next one mentioned in the bracket: position 3;

Position 3 moves to the next one mentioned in the bracket: position 2;

and Position 2, being the last appearing in the bracket, moves to the first one mentioned in the bracket: position 1.

Because positions 4, 5 and 6 are not mentioned, they remain unmoved."

The move R is an example of a move in which it is necessary to be careful about parity, since it does not take care of itself in the way it does for T. The table is:

Block in position	moves to	position
1	$\longrightarrow$	1
2	$\longrightarrow$	2
3	$\longrightarrow$	$\bar{5}$
4	$\longrightarrow$	$\bar{3}$
5	$\longrightarrow$	4
6	$\longrightarrow$	6

<u>Edge moves under R</u>

Bars have appeared above the 5 and the 3 in the last column because, for example, the first named face of position 3 (the green face of the green/red position) has moved to the <u>second</u> named face of position 5 (the yellow face of the green/yellow position) and by the same token the second face of position 3 has moved to the first face of position 5.

Once again the table may be abbreviated, either to

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & \bar{5} & \bar{3} & 4 & 6 \end{pmatrix}$$

or alternatively to

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 3 & 4 & 6 \end{pmatrix} F_{35} \text{ or } (354)F_{35},$$

where F_{35} means "flip the blocks in positions 3 and 5". Notice that we permute or "rearrange" by (354) first and <u>then</u> flip by F_{35}. We will see later that $F_{35}(354)$, flipping first and then rearranging, gives an entirely different pattern!

The reader is invited to find similar symbolism for the effects of L and B on the edge blocks.

With the notation we now have it is possible to write down any pattern on the popular pyraminx in terms of rearrangements from a pure pyraminx. For example one version of the pattern <u>Doubles</u> is:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \bar{4} & 2 & 6 & \bar{1} & 5 & 3 \end{pmatrix} = (14)(36)F_{14} .$$

Again *Nefertiti's Fan* arranges the edge blocks as:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \bar{4} & \bar{2} & 6 & \bar{1} & \bar{5} & 3 \end{pmatrix} = (14)(36)F_{14}F_{25}$$

and the vertices and middle blocks by

$$J'_t J'_\ell J'_r J'_b$$

so that the pattern as a whole is described by:

$$(14)(36)F_{14}F_{25}J'_t J'_\ell J'_r J'_b .$$

Every pattern can be described as a permutation of edge blocks followed by flips of some of the edges if necessary and then Jewels and vertex twists to complete the pattern. One more example: *Desert Palms* has the description:

$$(156)(234)F_{23}J'_b V_b .$$

(234) shows the cycling of the edge blocks round the red face and (156) is the effect of the full twist of the back (yellow/green/blue) vertex on the edge blocks. Of course this description of the pattern *is* only a description, and does not in itself give much help towards actually obtaining the pattern. One has to know *how* to cycle the edge blocks round the red face, for instance.

3. Calculating the Effect of Moves

Having methods of recording what moves are being made and what patterns are being shown is only half the battle. In order to make calculations it is ~~xxxx~~ necessary to marry moves to patterns.

Given a pyraminx, one can find by experiment what pattern appears from a given sequence of moves. The more interesting problem is to discover what sequence of moves will produce a given pattern. Related but more challenging questions ask for the minimal number of moves needed to produce a given pattern and the maximum number of moves a pattern can possibly be away from a pure pyraminx. The latter problem is unsolved in the general case. What is the maximum number of moves *ever* needed to return a pyraminx to pure form as quickly as possible? The answer lies somewhere between 12 and 30, but just where is one of the unresolved secrets of the pyraminx. By this time the reader will probably have devised a general solution which uses about 30 moves. An argument giving the lower limit follows.

Vertex twists never affect the pattern formed by the edge and middle blocks. Excluding the variations due to the vertex blocks, there are 75 582 720 ÷ 81 = 933 120 patterns. These can be formed by sequences of full twists. The first move to be made could be any one of eight possibilities (one of four vertices moved either clockwise or anticlockwise). The second and subsequent moves are chosen from six possibilities each, since one would not, when working efficiently, move the same vertex twice in succession. Now the maximum number of patterns which can possibly arise

from	2	moves	is	8×6	$=$ 48
from	3	moves	is	8×6^2	$=$ 288
from	4	moves	is	8×6^3	$=$ 1728
. . . . .					
from	7	moves	is	8×6^6	$=$ 373 248.

Of course at each stage fewer patterns may be possible as different sequences lead to the same pattern. But even if all the patterns obtained by sequences of 7 moves are different, there are still not enough to cover all the patterns known to exist. At least 8 moves are therefore needed. There are 2 239 488 sequences of 8 moves, enough to cover all the 933 120 patterns if there are not too many repeats. Adjusting the vertices after a sequence of 8 moves may add as many as 4 more moves (vertex twists) giving 12 moves in all.

The set of all sequences of moves of the pyraminx is called the <u>group</u> of the pyraminx, or the <u>pyraminx group</u>. Notice we are talking about <u>sequences</u> of moves (composite moves, or patterns, if one wishes to think of them this way). Two sequences are thought to be equal if they give rise to the same pattern. A "group" in this sense is a special kind of algebraic structure of which the pyraminx group is an example. Here are some ~~of the~~ facts about ~~it~~ the pyraminx group which may appear rather obvious but which are precisely what makes useful calculation possible:

(a) if one sequence of moves in the group, M_1 say, is followed by a second sequence, M_2, then the result is a third sequence, call it M_3, also in the group. We write $M_3 = M_1M_2$. For example

TL'TL followed by L'RLR' gives

TL'TLL'RLR' $=$ TL'TRLR'

(the L at the end of the first sequence and the L' at the beginning of the second have effectively cancelled each other out);

(b) If M_1, M_2 and M_3 are three sequences of the group then, for the purposes of calculation, $M_1(M_2M_3) = (M_1M_2)M_3$;

(c) the group contains series such as J_tJ_t' which leave everything unchanged; and

(d) every sequence has a reverse or <u>inverse</u> series which brings the pyraminx back to its pure state.

There are two or three equations and techniques of manipulation which are basic for calculation in the pyraminx group:

(i) The V's and the J's

Experiment and/or a little thought will show that if M is any sequence of moves of the pyraminx then

$$MV_i = V_i M \quad \text{and} \quad MJ_i = J_i M \quad \text{for } i = r, \ell, t \text{ or } b.$$

We say that the V_i and J_i commute with all other moves in the group. So to their inverses, the V_i' and J_i'. (This can be seen by taking the first equation, for example, and preceding and following both sides by V_i'.)

(ii) Flip followed by permutation

Suppose p is the permutation (235). Since the block in position 2 goes to position 3 under p, write $2^p = 3$. Similarly write $1^p = 1$, $3^p = 5$ and so on.

In general if p is any permutation of $1, \ldots, 6$ then n^p is to denote the position to which the block in position n goes under p. With this notation we claim that

$$F_{ij}\, p = p\, F_{i^p j^p}\,. \qquad (*)$$

For example $\qquad F_{25}(234) = (234)F_{35}\,,$

(F_{25} has become F_{35} as it moves across (234) because under (234) 2 goes to 3 and 5 goes to 5).

To see how this happens follow through what happens to the red/blue block in position 2 under $F_{25}(234)$:

> red/blue flips to blue/red and then the whole block moves to position 3 (the green/red position) so that the blue face moves to the green of position 3 and the red face moves to the red of position 3, i.e. red/blue goes to red/green, or in other words $2 \longrightarrow \bar{3}$.

Similarly $3 \longrightarrow 4$, $4 \longrightarrow 2$, $5 \longrightarrow \bar{5}$ and $6 \longrightarrow 6$ so that the final effect is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & \bar{3} & 4 & 2 & \bar{5} & 6 \end{pmatrix} = (234)\, F_{35}$$

as required.

A bit of juggling with symbols and the general equation $(*)$ can be shown in the same way.

(iii) Permutation followed by permutation

Suppose the sequence of moves M_1 has the effect of permuting the edge blocks by $p_1 = (12)(3456)$ leaving all vertices and middle blocks unchanged, and that M_2 permutes the edge blocks by

$P_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6 & 4 & 5 \end{pmatrix} = (132)(465)$. To answer the question "what is the effect of M_1 followed by M_2 we once again simply follow through what happens to each block in turn. M_1 followed by M_2 subjects the blocks to the sequence of permutations:

$$(12)(3456)(132)(465).$$

In the sequence

1 moves to 2 by (12), and then
2 is left fixed by (3456), and then
2 moves to 1 by (132), and then
1 is left fixed by (465).

So when all is told, 1 stays where it is. Doing the same thing for 2:

2 goes to 1 which goes to 1 which goes to 3 which goes to 3

so that all told 2 moves to 3. A quicker way of writing this sequence would be:

$$2 \longrightarrow 1 \longrightarrow 1 \longrightarrow 3 \longrightarrow 3.$$

Similarly for 3:

$$3 \longrightarrow 3 \longrightarrow 4 \longrightarrow 4 \longrightarrow 6,$$

i.e. 3 goes to 6 in the end, and in the same way 6 goes to 2 and 4 and 5 stay where they are. The total effect is therefore

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 6 & 4 & 5 & 2 \end{pmatrix} = (236),$$

in other words $(12)(3456)(132)(465) = (236)$.

The above is an example of multiplying two permutations together. The same basic technique may be used for any pair of permutations.

(iv) Two useful permutation relationships

The "inverse" of a permutation cycle $(i\ j\ \ldots\ s\ t)$ is the reverse cycle $(t\ s\ \ldots\ j\ i)$. Thus, for example,

$$(23654)(45632)$$

leaves everything unchanged. (45632) is the inverse of (23654).

If c_1c_2 is a product of two cycles then the inverse is $c_2^{-1}c_1^{-1}$ where c_1^{-1}, c_2^{-1} are the reverse cycles of c_1 and c_2 respectively. Notice the change of order in the inverse. For example:

$$(132)(256)(652)(231)$$

leaves everything fixed and so (652)(231) is the inverse of (132)(256). The first factor of the inverse, (652), is itself the inverse of the second factor of the product, and likewise the second factor of the inverse is the inverse of the first factor of the original product.

Another useful formula for calculating certain products of permutations is the following:

if p is any permutation and $(i\ j \dots s\ t)$ is a cycle, then

$$p^{-1}\ (i\ j \dots s\ t)\ p = (i^p\ j^p \dots s^p\ t^p).$$

For example $(642)^{-1}(12345)(642) = (246)(12345)(642)$
$= (16325)$.

The 2 and 4 of the original cycle (12345) have become 6 and 2 respectively as these are their images under the permutation (642). This technique is well worth remembering when dealing with the pyraminx group, where moves of the form $p^{-1}q^{-1}p\ q$ are very common. Such moves only affect a few blocks and leave most of them unchanged, a very useful property.

4. Examples of move sequences

Now it is possible to write down the effect both of the moves T, R, L, B, T' etc., and of sequences made from them. Check that

$T = (132)\ J_t$	$T' = (123)\ J_t'$
$R = (354)\ F_{35}J_r$	$R' = (345)\ F_{34}J_r'$
$L = (246)\ F_{26}J_\ell$	$L' = (264)\ F_{46}J_\ell'$
$B = (165)\ J_b$	$B' = (156)\ J_b'$.

A number of sequences are worth mentioning either because they are particularly useful or because they illustrate how to use the mechanism which has been set up for calculation (and often for both reasons):

(i) $$\begin{aligned} BT'B'T &= (165)J_b(123)J_t'(156)J_b'(132)J_t \\ &= (165)(123)(156)(132)J_bJ_b'J_t'J_t \\ &= (135). \end{aligned}$$

This sequence simply rotates the edge pieces round the green face. Reversing the rotation using the R and B vertices instead of the B and T ones:

$$\begin{aligned} B'RBR' &= (156)(345)F_{35}(165)(345)F_{34}J_b'J_bJ_rJ_r' \\ &= (156)(345)(165)(345)\ F_{14}F_{34} \\ &= (153)\ F_{14}F_{34} \ . \end{aligned}$$

Now putting the two together:

$$BT'B'TB'RBR' = (135)(153)\ F_{14}F_{34} = F_{14}F_{34}\ ,$$

a Cat's Paws in eight moves.

(ii) On the other hand repeating the first rotation of (i) using the T and R vertices instead of the B and T vertices gives:

$$\begin{aligned} TR'T'R &= (132)\ J_t\ (345)\ F_{34}\ J_r'\ (123)\ J_t'\ (345)\ F_{35}\ J_r \\ &= (132)(345)(123)(354)\ F_{13}\ F_{35} \quad = (135)F_{15} \end{aligned}$$

and putting this together with the first rotation:

$$BT'B'TTR'T'R = BT'B'T'R'T'R$$
$$= (135)(135)\ F_{15} = (153)\ F_{15}\ .$$

Inspection shows this is Doors round the green face as base, in 7 moves.

(iii) $$TB = (132)\ J_t\ (165)\ J_b = (132)(165)\ J_tJ_b$$
$$= (13265)\ J_tJ_b\ .$$

Since J_t and J_b commute with all other moves

$$(TB)^n = (13265)^n\ J_t^n\ J_b^n$$

for any integer n. In particular

$$(TB)^3 = (13265)^3\ J_t^3\ J_b^3 = (13265)^3 = (16352)$$

and only edge pieces are affected by this power of TB. Again

$$(TB)^5 = (13265)^5\ J_t^5\ J_b^5 = J_t^2\ J_b^2 = J_t'\ J_b'\ .$$

If we did not know it before, we have now calculated a sequence of 10 moves giving the Jewel pattern on two vertices. From it can be deduced a sequence giving Jewel on just a single vertex, although this will not be a minimal sequence.

5. A Theorem about impossible patterns

Every permutation can be written as a product of cycles of length two. For example

$$(135) = (13)(15)$$
$$(135)(2436) = (13)(15)(24)(23)(26)$$

and so on.

An even permutation is one which can be written as a product of an even number of cycles of length two. Thus (135) is an even permutation, and so are (13524) and (135)(562). It is not difficult to work out that:

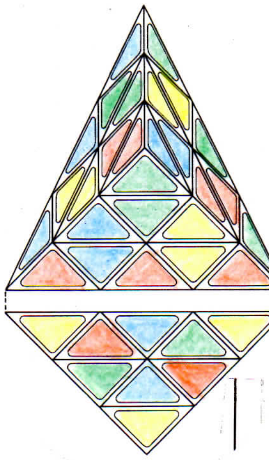
(i) the product of two even permutations is again an even permutation;

(ii) if a product of cycles of length two is in fact the identity (leaves everything fixed) then it has an even number of factors;

(iii) no even permutation can be written as a product of an odd number of cycles of length two. (If a permutation is not even, then it is odd, and vice versa.)

Looking at the expressions for T, R, L and B we see that they all involve even permutations of the edge blocks, and only even permutations. It follows that it is not possible to have a pyraminx pattern which implies an odd permutation of the edge blocks.

Considering the effect of T, R, L, and B on the facelets it is also possible to prove that an odd permutation of the facelets is impossible. In particular, therefore, we can be sure it is impossible to "flip" a single edge piece or to exchange two edge pieces while leaving all the others fixed.

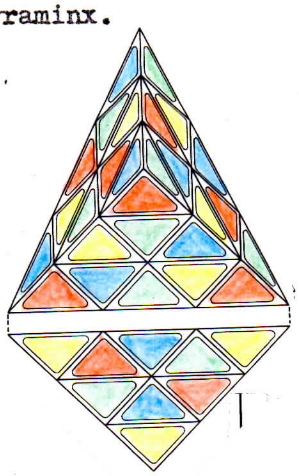


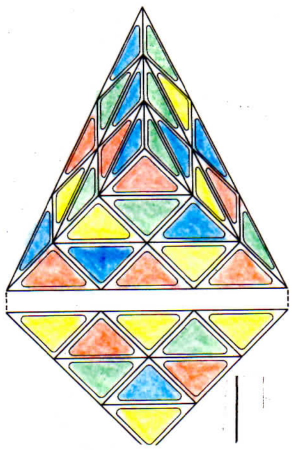
For example the above pattern is convincing and enjoyable. But is it possible? The formula for the pattern is:

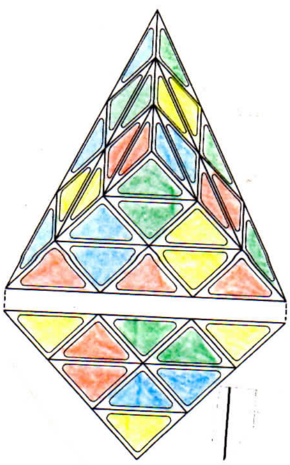
$$(1\ 6\ 2\ 4\ 3\ 5)\ F_{23}\ F_{46}\ J_b J_\ell J_r J_t V'_b V'_\ell V'_r\ .$$

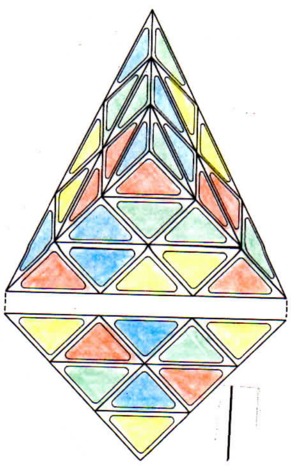
The permutation $(1\ 6\ 2\ 4\ 3\ 5) = (16)(12)(14)(13)(15)$ and is therefore odd. The pattern is impossible.

The reader may like to find the formulae for the following patterns and deduce whether or not they are possible on the popular pyraminx.









6. The Pyraminx Group in the Abstract

Those who already have some acquaintance with abstract groups may recognise that the group of the popular pyraminx is the direct product

$$P = E \times B$$

of an elementary group B of order 3^8 and the "edge group" E of moves of the edge pieces.

The subgroup B is

$$\langle J_t \rangle \times \langle J_\ell \rangle \times \langle J_r \rangle \times \langle J_b \rangle \times \langle V_t \rangle \times \langle V_\ell \rangle \times \langle V_r \rangle \times \langle V_b \rangle .$$

The edge group E is an extension of an elementary group

$$F = \langle f_1 \rangle \times \langle f_2 \rangle \times \; \dots \; \times \langle f_5 \rangle$$

of order 2^5 by the alternating permutation group A_6. To show how A_6 acts on F (in the holomorph of F) it is best to think of A_6 as being the even permutations of the edge blocks in positions $1,2,\cdots,6$ as before, and then the elements $f_1, f_2, \dots, f_5$ as representing Cats Paw flips such that $f_i = F_{i\,i+1}$, $i = 1,\dots,5$.

Denote the block in position i by E_i. Now the relationship between the E's and the f's may be shown by the schema:

$$\begin{array}{ccccccccccc} E_1 & & E_2 & & E_3 & & E_4 & & E_5 & & E_6 \\ & \diagdown \; f_1 \; \diagup & & \diagdown \; f_2 \; \diagup & & \diagdown \; f_3 \; \diagup & & \diagdown \; f_4 \; \diagup & & \diagdown \; f_5 \; \diagup & \end{array}$$

All the F_{ij} may be expressed in terms of the f's, reading off from this diagram. Thus for example

$$F_{25} = f_2 f_3 f_4 = h, \text{ say} .$$

(E_3 and E_4 are each flipped twice by h, and hence are left invariant by it.)

We have already seen, in effect, that for an arbitrary p in A_6 $p^{-1} f_i p$ flips E_{ip}. This is the appropriate form of equation (*) for the present discussion (see paragraph 3(ii) above). The action of p on the group F is now calculated as follows:

under p,

$$\begin{array}{ccccccccccc} E_1 & & E_2 & & E_3 & & E_4 & & E_5 & & E_6 \\ & \diagdown \; f_1 \; \diagup & & \diagdown \; f_2 \; \diagup & & \diagdown \; f_3 \; \diagup & & \diagdown \; f_4 \; \diagup & & \diagdown \; f_5 \; \diagup & \end{array}$$

becomes

$$\begin{array}{ccccccccccc} E_{1p} & & E_{2p} & & E_{3p} & & E_{4p} & & E_{5p} & & E_{6p} \\ & \diagdown \; g_1 \; \diagup & & \diagdown \; g_2 \; \diagup & & \diagdown \; g_3 \; \diagup & & \diagdown \; g_4 \; \diagup & & \diagdown \; g_5 \; \diagup & \end{array}$$

where the g_i are the appropriate elements of F, g_i flips E_{ip} and $E_{(i+1)p}$. For example suppose $p = (164)$. Now the resulting diagram is

$$E_6 \quad g_1 \quad E_2 \quad g_2 \quad E_3 \quad g_3 \quad E_1 \quad g_4 \quad E_5 \quad g_5 \quad E_4$$

and g_1 flips E_6 and E_2, i.e. $g_1 = f_2f_3f_4f_5$,
g_2 flips E_2 and E_3, i.e. $g_2 = f_2$,
g_3 flips E_3 and E_1, i.e. $g_3 = f_1f_2$,
g_4 flips E_1 and E_5, i.e. $g_4 = f_1f_2f_3f_4$ and
g_5 flips E_5 and E_4, i.e. $g_5 = f_4$.

In other words

$$(164)^{-1}f_1(164) = f_2f_3f_4f_5,$$
$$(164)^{-1}f_2(164) = f_2 \quad \text{and so on.}$$

With this understanding of how A_6 is to act on F, the edge group E is the subgroup A_6F of the holomorph of F. The reader may prove that $E = P'$, the derived group of the pyraminx group, and in fact $E = E'$. The centre of P is just the subgroup B.

In group theoretic terms the pyraminx puzzle is now the problem of taking a rather arbitrary seeming set of 8 generators of P, namely V_t, V_ℓ, V_r, V_b, $(132)J_t$, $(246)f_2f_3f_4f_5J_\ell$, $(354)f_3f_4J_r$ and $(165)J_b$, and then trying to express every element of P in terms of these generators in the most economical way.